Game Theory: Problem Set 1

1 Basic understanding

For both of the games below, identify: (i) dominant strategies, (ii) dominated strategies, and (iii) Nash equilibria.

|  | C | D |
| --- | --- | --- |
| A | 5,2 | **9**,**3** |
| B | 1,12 | 6,0 |

Answer -

I. A is a dominant strategy

Ii. B is a dominated strategy

Iii. A,D is a Nash equilibrium

|  | L | M | R |
| --- | --- | --- | --- |
| U | 1,0 | 2,1 | **3**,**3** |
| D | **3**,**6** | 2,2 | 1,4 |

Answer -

I. No dominant strategies

Ii. No dominated strategies

Iii. D,L & U,R are Nash equilibria

2 Simple advertising game

There are two firms and ten consumers. Each sale to a consumer generates $1 in profits. All ten consumers buy from one firm or the other. Each firm chooses whether to advertise or not. Advertising costs $4. If both firms choose the same level of advertising (i.e., both choose to advertise or both choose not to advertise), they split the ten consumers evenly. If one firm advertises and the other one does not, the advertising firm gets all ten consumers. You can assume that the firms payoff (utilities) simply equal their profits. Write this game in a matrix form and find all Nash equilibria. Is this game one of the games we have studied thus far in class?

Total profit through sales = $10

Advertising costs per firm = $4

Ads/Ads profit per firm = $5 (splitting the 10 customers) - $4 = $1

Ads/No Ads profit = $10 (all 10 customers) - $4 = $6

No Ads/No Ads profits = $5 (splitting the 10 customers)

|  | Ads | No Ads |
| --- | --- | --- |
| Ads | **1,** **1** | **6**, 0 |
| No Ads | 0, **6** | 5, 5 |

Answer - Nash equilibria = Ads/Ads (1,1)

This is a symmetrical game (similar to the prisoners’ dilemma or stag game) which we have seen before

3 Splitting the pie

Player 1 and 2 are bargaining over how to split $10. Each player i names an amount si, between 0 and 10, for herself. These numbers do not have to be in whole dollar units; in other words si ∈ [0, 10]. The two choices are made simultaneously. Each player’s payoff is equal to the dollar amount she obtains. We will consider two possible games the players might play.

3.1 Excess demands destroy the whole pie

If s1 + s2 ≤ 10, then player 1 gets s1 and player 2 gets s2. (If s1 + s2 < 10, the remaining 10 - s1 - s2 is destroyed.) If, however, s1 + s2 > 10, then both players get zero and all of the money is destroyed. What are the Nash equilibria of this game?

Answer - The Nash equilibria would be any combination in which s1 + s2 = 10

3.2 Allocation respects humility

Now consider an alternate game. As before, if s1 + s2 ≤ 10, then player 1 gets s1 and player 2 gets s2. (If s1 + s2 < 10, the remaining 10 - s1 - s2 is destroyed.) Now, however, if s1 + s2 > 10, the player who named the smaller amount gets that amount and the other person gets the remainder of the money. For example, if s1 + s2 > 10 and s1 < s2, then u1 = s1 and u2 = 10 - s1. If s1 + s2 > 10 and it happens that s1 = s2, then both players get $5, i.e., u1 = u2 = 5. What are the Nash equilibria of this game?

Answer - The Nash equilibria would be (5,5), (6,5), (5,6), and (6,6)

For (5,5) we would see s1 + s2 ≤ 10 and each player gets u = 5

For (6,5) and (5,6) we would see s1 + s2 > 10 and s1 < s2 (5 < 6) so u1 = 5 and u2 = 10 - 5 = 5 so each player would get u = 5

For (6,6) we would see s1 + s2 > 10 and it happens that s1 = s2 so each player would get u = 5

4 Competing in quantities

Suppose two firms selling an identical product engage in Cournot competition. There are 100 potential customers and the industry demand is 100 - p. Firms choose quantities qi ∈ [0, 50], which leads to a price that equalizes supply and demand. Firms maximize their profits. This applies to all four problems below.

4.1 Baseline game

Suppose the firms choose their quantities simultaneously. Each firm’s marginal cost is $10. Find the Nash equilibrium of this game.

Answer - [30, 30]

Qd = 100-p

Qd = Qa + Qb

MC = 10

100 customers

Q1= [0,50]

Πa = Qa[100 - Qa-Qb]- 10Qa

Πa = 100Qa-Qa^2 - QaQb-10Qa

Derivative = 100-2Qa - Qb – 10

= 90 - 2Qa - Qb

Set equal to 0 to max profit

0 = 90 - 2Qa - Qb

**Qa = 90-2Qb**

Πa and Πb are equal

Πb = Qb[100 - Qa-Qb]- 10Qb

Πb = 100Qb-Qb^2 - QaQb-10Qb

Derivative = 100-2Qb - Qa – 10

= 90 - 2Qb - Qa

**Qb = 90 - 2Qa**

Qa = 90 - 2(90 - 2Qa)

Qa = 90 - 180 + 4Qa

Qa = - 90 + 4Qa

-3Qa = - 90

3Qa = 90

**Qa = 30**

**Qb = 30**

4.2 Raising rival’s costs

Now, suppose that firm 1 can spend $X to increase the other firm’s marginal cost to $20 (leaving its own marginal cost unchanged). Then, given the new costs, the firms choose their quantities simultaneously. What is the most money (i.e., the largest X), that firm 1 would be willing to spend on this?

Answer X = 511

X = amount Firm 1 would pay to increase firm 2’s MC to $20

Πa = Qa [ 100-Qa-Qb] - 10Qa - X

= 33.33 (100 - 2

Πb = Qb[100-Qa-Qb] - 20Qb

= 30 (40) - 300

1200-300

=900

1444.5

566

Πa = Qa100-Qa^2 - Qb - 10Qa - X

Derivative = 100-2Qa - Qb - 10 - X

Set to 0

0 = 90 - 2Qa - Qb - X

Qb = 90 - 2Qa

Πb = Qb100-Qb^2 - QaQb - 20Qb

Derivative = 100-2Qb-Qa - 20

0 = 80 - 2Qb - Qa

Qa = 80 - 2Qb

Plug

Qb = 90 - 2 ( 80-2Qb )

Qb = 90 - 160 + 4Qb - X

OR

Qb = 90 - 2Qa

Qb = 90 - 2(80-2Qb)

Qb = 90-160+4Qb

Qb = -70-4Qb

3Qb = 70

Qb = 70/3

70/3 = 90 - 2Qa

**Qb=23.33**

23.33=90-2Qa

66.67=2Qa

**Qa = 33.33**

Πb = Qb[100-Qa-Qb] - 20Qb

= 30 (40) - 300

1200-300

=900

Solve

Πa = Qa [ 100-Qa-Qb] - 10Qa

= *33.3[100-33.3-70/3] - 10\*33.33*

*= 1411*

***X = 1411 - 900***

***= 511***

4.3 Covert information

As in the baseline game, each firm’s marginal cost is $10. Suppose that, unbeknownst to firm 2, firm 1 can observe the quantity chosen by firm 2 before it chooses its own quantity. (For example, firm 1 has a spy in firm 2’s headquarters and firm 2 does not suspect this is even possible.) How do the profits of each firm change? Along with an exact numerical answer, provide an intuition for the direction of the change(s), if any.

If firm 1 knew that firm 2 was choosing a quantity of 30, they would not change their quantity because (30, 30) is the nash equilibrium for this scenario.

Answer - Profits would not change for this scenario and would hold at $900

Πb = Qb[100-Qa-Qb] - 20Qb

= 30 (40) - 300

1200-300

=900

EX: Qa = 29

Πa = 29 [100 - 29-30]- 10(29)

Πa = 899

EX: Qa = 31

Πa = 31 [100 - 31 -30]- 10(31)

Πa = 899

4.4 Overt information (difficult; prelude to week 3 material)

As in the baseline game, each firm’s marginal cost is $10. Suppose that everyone knows that firm 1 can observe the quantity chosen by firm 2 before it chooses its own quantity. How do the profits of each firm change? Along with an exact numerical answer, provide an intuition for the direction of the change(s), if any.

Qd = 100-P

P = 100 - Qd (inverse demand curve)

Qd = Qa + Qb

MC = 10

Πb = Qb[100-Qa-Qb] - 10Qb

Πa = Qa[100-Qa-Qb] - 10Qa

R2(Qb) = Firm A’s reaction to Firm B’s decision

Stackleberg profits considering Firm B chooses their quantity first

First find the output that maximizes the “followers” (Firm A) profit:

Πa = Qa[100-(Qa+Qb)] - 10Qa

= 100Qa - Qa^2 - QaQb - 10Qa

= 90Qa - Qa^2 - QaQb

Derivative:

0 = 90 - 2Qa - Qb

Qa = 90-Qb/2

Now best response function of the “leader” (Firm B)

Πb = Qb[100-(Qb+Qa)] - 10Qb

Πb = Qb[100-(Qb+Qa(Qb))] - 10Qb

Πb = Qb[100-(Qb+90+Qb/2)] - 10Qb

Πb = (100Qb -Qb^2 + 10Qb)/2 - 10Qb

0 = (100 - 2Qb + 10)/2 - 10

10 = 90-2Qb/2

20 = 90-2Qb

Qb = 45

Qa = 90-Qb/2

Qa = 90-45/2 = 22.5

Πa = Qa[100-(Qa+Qb)] - 10Qa

Πa = 22.5(100-(22.5+45)) - 10(22.5)

**Πa = 731.25 - 225 = 506.25**

Πb = Qb[100-(Qb+Qa)] - 10Qb

Πb = 45[100-(45+22.5)] - 10(45)

**Πb = 1462.5 - 450 = 1,012.5**

**Because Firm B makes their decision and Firm A responds, Firm B can choose a quantity that produces a larger profit for the Firm and Firm A must respond with their “best response” which in this case is going to lead to an increase in profits for Firm B and a decrease in profits for Firm A in comparison to if they were to choose the Nash equilibrium.**